Day 3

- Counting and its use (数え方とその利用）
  - Double counting
    二重数え上げ法
    Conceptually, a nother view of pigeonhole principle, where we count pigeons in two ways.
    鳩の巣原理を見方を変えたようなもの。

- Problem 1: Is there a graph with 5 vertices such that each vertex degree is 3? If yes, construct it.

5頂点のグラフで、全ての頂点次数が3であるグラフは存在するか？存在するのなら構成せよ。

Solution by double counting

- Prove by counting same set in two ways
  - 同じものを2通りに数えることで証明する
- Count the pair (v,e) of vertex and edge in two ways.
  - 頂点と辺の対(v, e): 頂点vは辺eの端点を2通りに数える

Sum of numbers of divisors

Problem　２：　Let f(m) be the number of divisors of a natural number m.

Let G(n)=（1/n）$\sum_{1 \leqq m \leqq n}$f（m）.

Compute G(1024) (within error 2)

- 自然数ｍの約数の数をf（ｍ）とする。

G(n)=（1/n）$\sum_{1 \leqq m \leqq n}$f（ｍ）としたとき、

G(1024)を(誤差２の範囲で）求めよ

Sum of numbers of divisors

$n\,G(n)=\sum_{1\leqq m\leqq n} f(m)$.

This is the number of pairs (m, a) such that a divides m.

Count the same number from a, then this is the pair (ab, a) such that ab $\leqq$ n

There are [n/a] such pairs.

Thus, $nG(n) = \sum_{1\leqq a\leqq n} [n/a]$. Thus, we have

$(\sum_{1\leqq a\leqq n} 1/a) - 1 < G(n) < \sum_{1\leqq a\leqq n} 1/a \sim \log n$

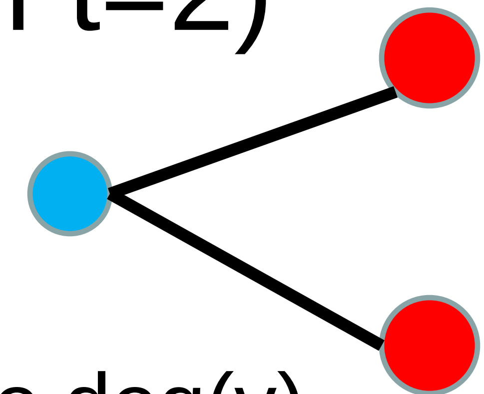
Extreamal graph theory

Consider a bipartite graph G = (V, W, E) such that |V| =k and |W|= n.

If the graph does not contain $K_{s,t}$ as a subgraph, for constants s and t, the number of edges is $O(k^{1-1/s}\, n)$

二部グラフ G = (V, W, E) において |V| =k , |W|= nとする。 もしこのグラフが $K_{s,t}$ を部分グラフに持たなければ、辺の数は $O(k^{1-1/t}\, n + k)$

What to count (if t=2)



- Count triples in G
- If the blue node v has degree deg(v), it creates deg(v)(deg(v)-1) such triples
- Each red pair contributes to at most s-1 triples
- Thus $\Sigma_{v \in V}$ deg(v)(deg(v)-1) < (s-1) n(n-1)
- Thus, k $(m/k)^2$ - m < (s-1) n(n-1)
- Thus, m < k+ $k^{1/2}$ $(s-1)^{1/2}$ n

Application to geometry

Given n points and m lines in the plane. Show that the point-line incidence is $O(n m^{1/2} + m)$.

平面上のn個の点とm本の直線を考えると、点と直線の隣接対の数は$O(n m^{1/2} + m)$.

Sperner system

- Given a set S of size n, consider a set F of subsets S. F is called a Sperner system if no pair in F is in the inclusion relation, that is, there are no pair A, B in F such that $A \subset B$
- Example, the set of all subsets of cardinality k is a Sperner system.
- Problem: Show that the largest Sperner system has cardinality $_{n}C_{[n/2]}$

Proof by counting

- Chain of sets: sequence of sets with inclusion relation
- Given a Sperner system F, count the number of pairs (A, C) such that A is in F and C is a chain containing A.
- User double-counting.

Chain and antichain

- Given a partially ordered set A, a chain of A is a sequence of elements

  $a(1) < a(2) < \ldots < a(k)$
- An antichain is a set of uncomparable elements of A.
- $\mu(A)$: size of maximum antichain
- $\tau(A)$: size of minimum cover of A by chains
- Dilworth's theorem: $\mu(A) = \tau(A)$
  - Easy : $\mu(A) \leqq \tau(A)$
  - The other direction is difficult

Another proof of Sperner's theorem

- If A is the set P(S) of all subsets of S, a Sperner system is an antichain
- We can prove Sperner's theorem from μ（A)≦τ（A)
  - It suffices to make a chain cover of size $_{n}C_{[n/2]}$