Day 5

- A Gem of Combinatorics
組合わせ論の宝石
  - Proof of Dilworth’s theorem
  - Some Young diagram combinatorics
  ヤング図形の組合せ論

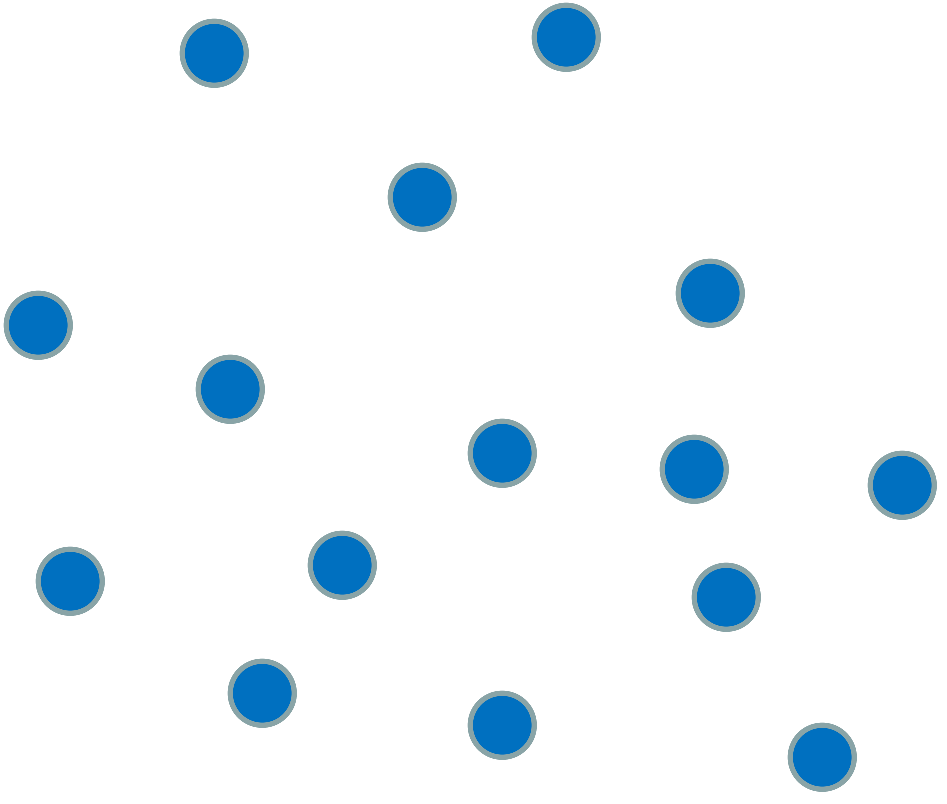
Recall the last lecture and try the homework solution

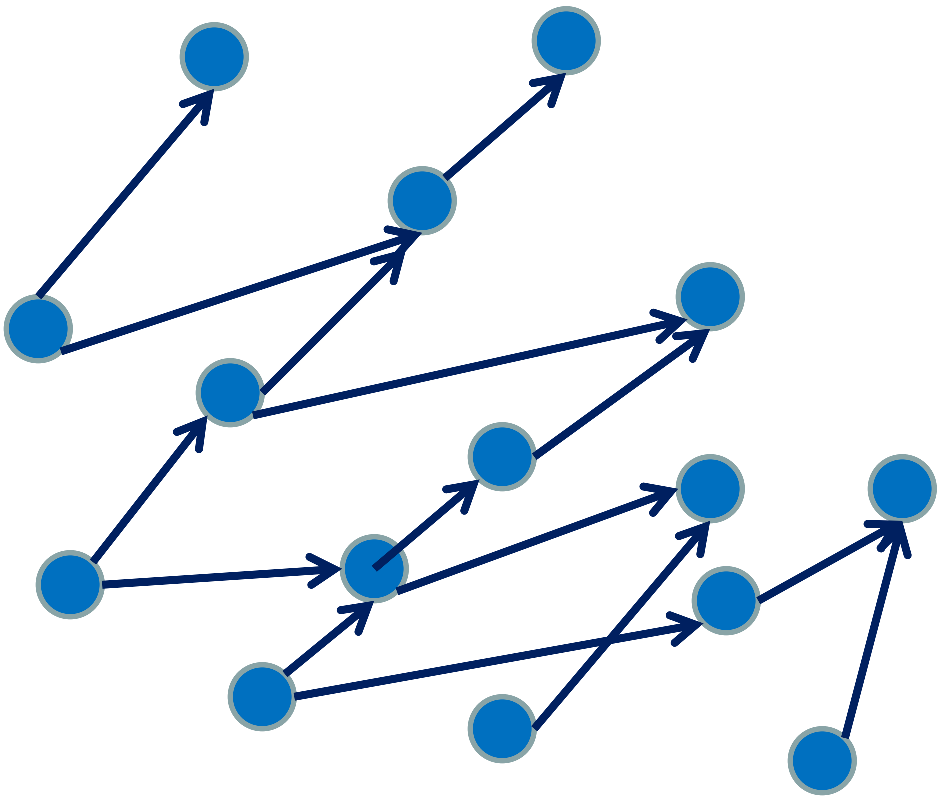
復習と宿題確認

Partially Ordered Set (poset)
半順序集合(ポセット）

- Given a set S, a relation > is called a partial order if it satisfies the following axioms
  - If x > y and y> z then x>z   (transitive)
  - x >y and y<x if and only if x=y (anti-symmetric)

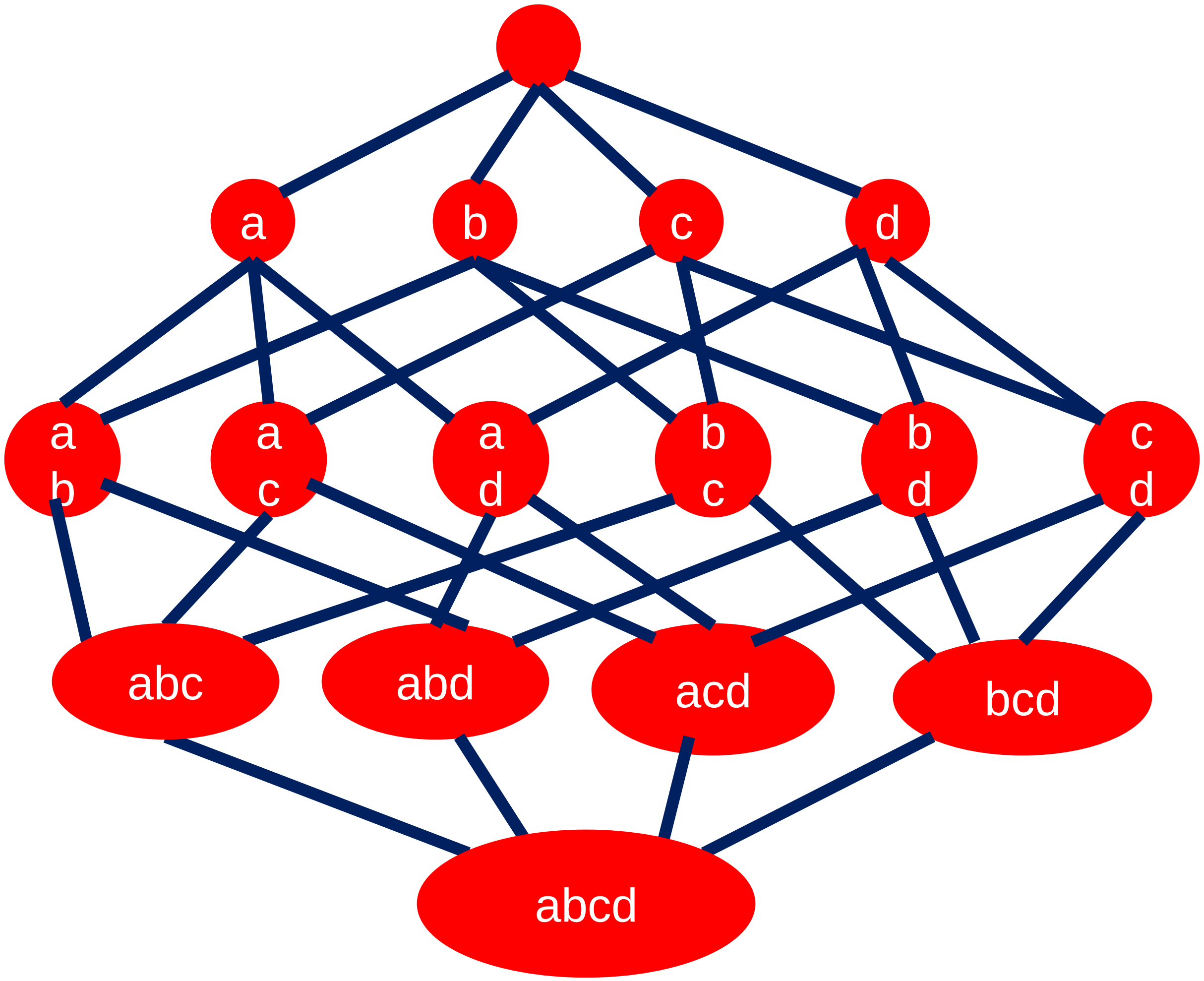
  推移律と反対称性を持つ関係を半順序、それを持つ集合を半順序集合と呼ぶ
- A set with a partial order is called a poset
  - Example 1. A set of numbers
  - Example 2. A set of points (in which order?)
  - Example 3. A set of subsets of a finite set S





Chain and antichain

- A chain in a poset **A** is a sequence of elements $a(1) < a(2) < \ldots < a(k)$
  - チェイン：全順序列のこと
- An antichain is a set of uncomparable elements of **A**.
  - 反チェイン：互いに比較不能な要素の集合
- μ(**A**): size of maximum antichain
- τ(**A**): size of minimum chain partition of **A**
- Exercise: show that μ(**A**)≦τ(**A**)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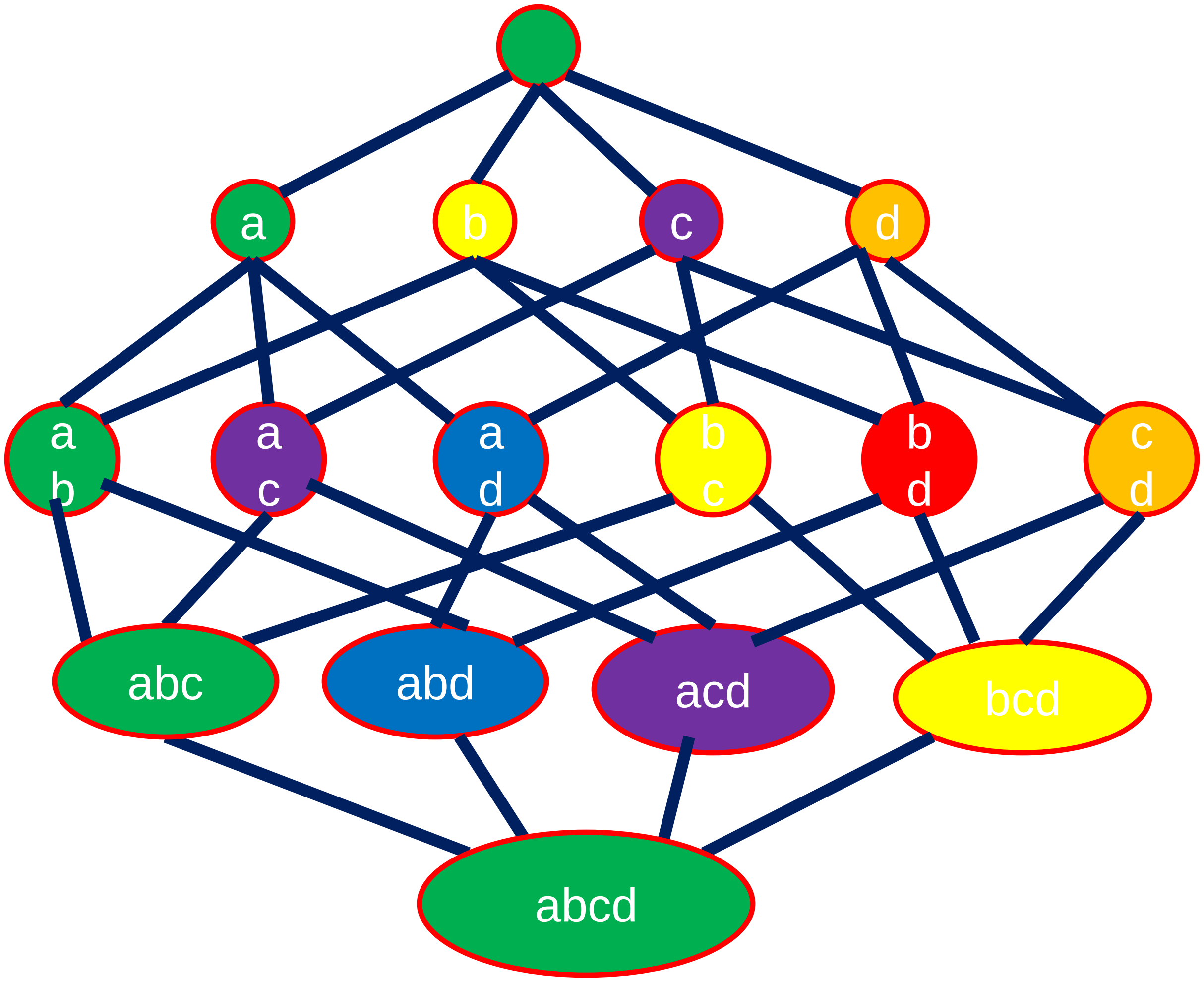
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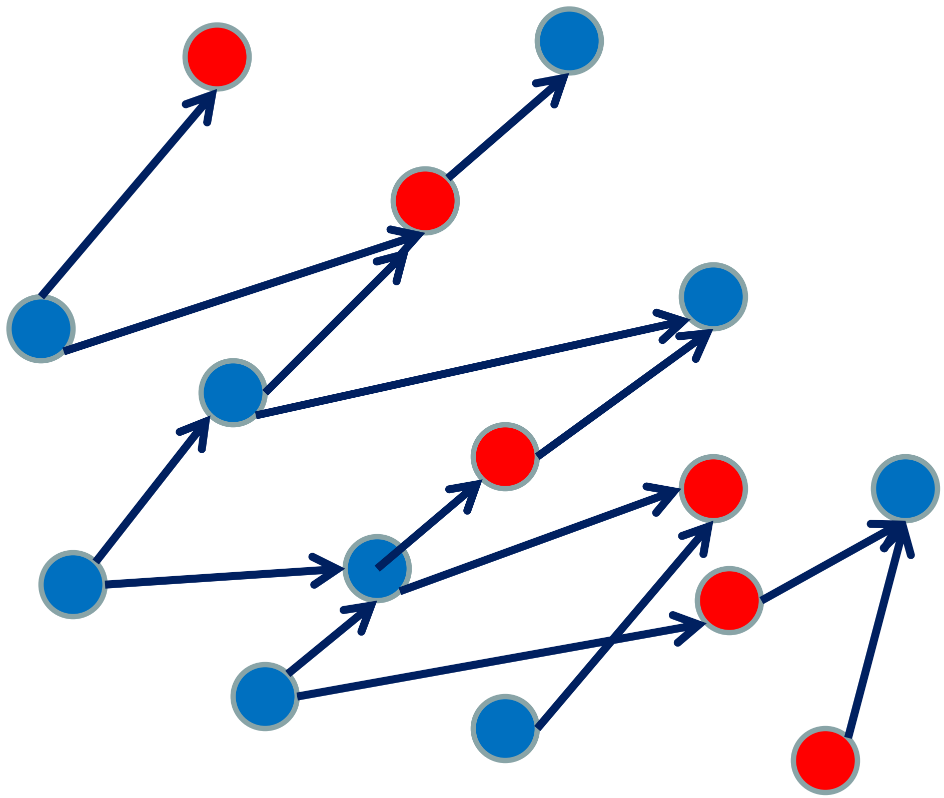


Dilworth's theorem ディルワースの定理

- Dilworth's theorem: μ（**A**)=τ（**A**)

A beautiful "duality" of chain and antichain

チェインと反チェインの間の双対性を示す美しい関係

Dilworth’s theorem

- Dilworth’s theorem: μ（**A**)=τ（**A**)

Applications:

- Covering-matching duality (done)
  被覆とマッチングの双対性
- Hall’s marriage theorem 結婚定理 (done)
  A Quiz on switching network スイッチ結合クイズ
- Revisit Erdos-Szekeles theorem
  - エルデシュ－セケレシュ定理再訪
- Theory of Young tableaux (without proofs)

Refined marriage theorem 結婚定理の精密化

- Let δ（A）＝|A|-|N(A)| and its maximum among all subsets A of V is δ. Then, the size of maximum matching is |V| - δ

δを|A|-|N(A)|の最大とすると、最大マッチングのサイズは|V|-δ

δ＝0なら結婚定理と一致

- Proof is an exercise

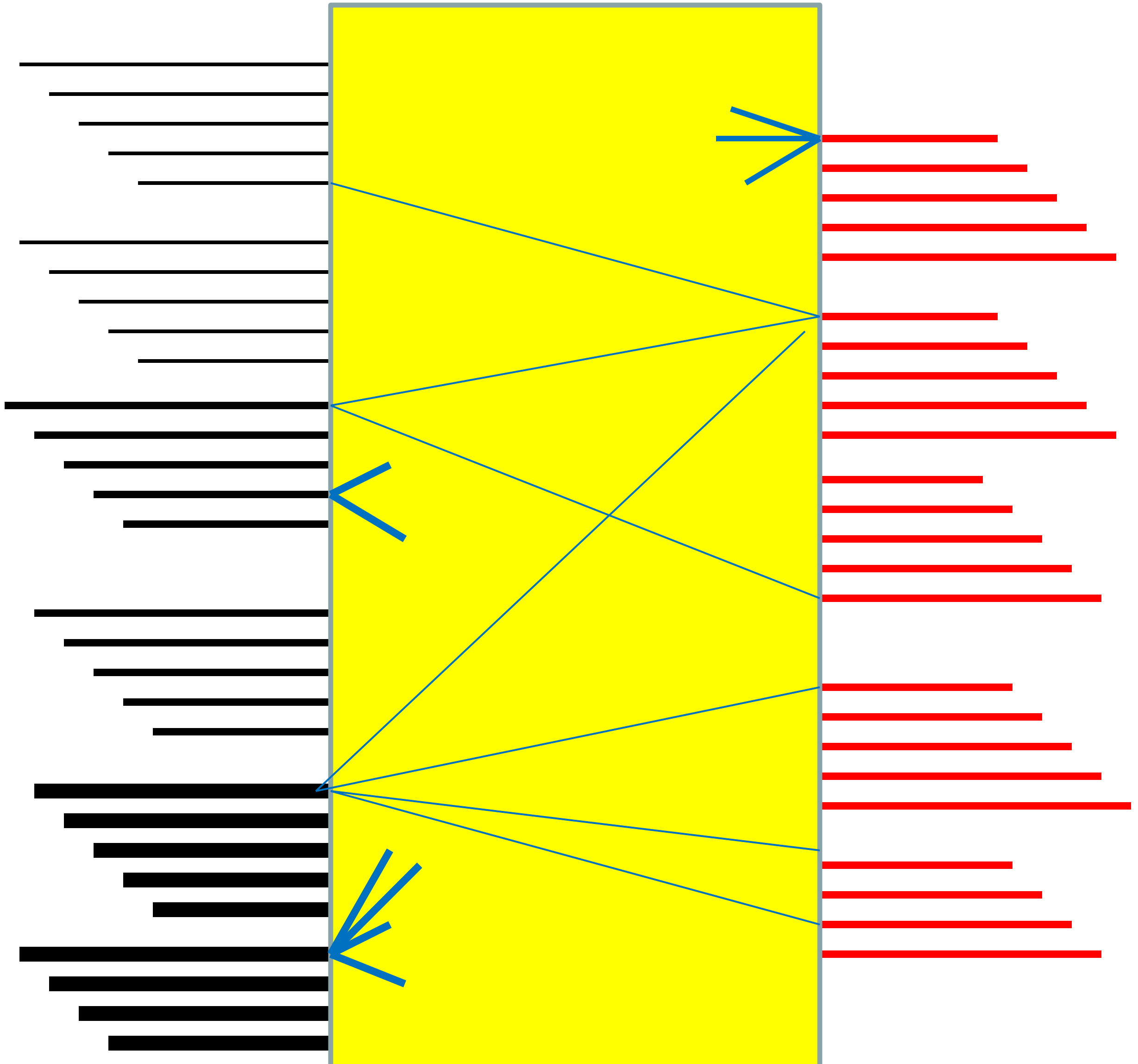
Quiz （配線問題）

- There are 30 input lines connecting to 24 output lines. 10 inputs connects to only one output line, 10 inputs to two lines, and 10 inputs to four lines.

  Each output line is connected to at most three input lines.

  Prove that, we can establish 20 parallel connection irrelevant to the way of connection.

Hint: Show that $\delta(A) \leqq 10$ for any A.



Proof of Dilworth’s theorem

- Dilworth’s theorem: μ（**A**)=τ（**A**)
- Definition: x ∈**A** is effective if there is an antichain of length μ(**A**) containing x.
- Lemma 1. Suppose that we have a chain partition M(1),M(2)..,M(k) of **A** for k=μ（**A**）, and let x(i) be the maximal effective element in M(i). Then, x(1),x(2),…,x(k) is an antichain.

μ（**A**)のサイズのチェイン分割があったとき、チェインの極大効果元たちはアンチチェインになる

Proof of Dilworth's theorem

- $\mu(\mathbf{A})=\tau(\mathbf{A})$
  - Show $\mu(\mathbf{A}) \geqq \tau(\mathbf{A})$ by induction 帰納法で証明

1 . If $|\mathbf{A}| = 1$, then easy

2. Remove a maximal element a from **A** to have **A**' $=\mathbf{A} - \{a\}$: $\mu(\mathbf{A}') = \tau(\mathbf{A}') = k$ by induction

3. Consider a maximum chain partition of **A**', and find x(1), x(2),..,x(k) of Lemma 1.

4. If a is not comparable to any x(i), fine (why?)

5. If a > x(1), -→ show at the blackboard.

What mathematicians do

- If two series of values are same, then, find an identity of functions.
- If two sets has the same cardinality, find a canonical one-to-one correspondence.

二つの集合が同じ大きさなら、自然な一対一対応を見つけよう

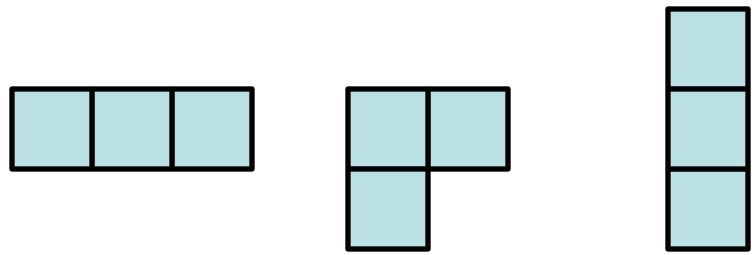
  - Cantor defined “cardinality” via one-to-one correspondence
    - |The set of integers | = |The set of rational numbers| since there is no one-to-one correspondence
    - |The set of integers| < |The set of real numbers|

Revisit Erdos-Szekeles

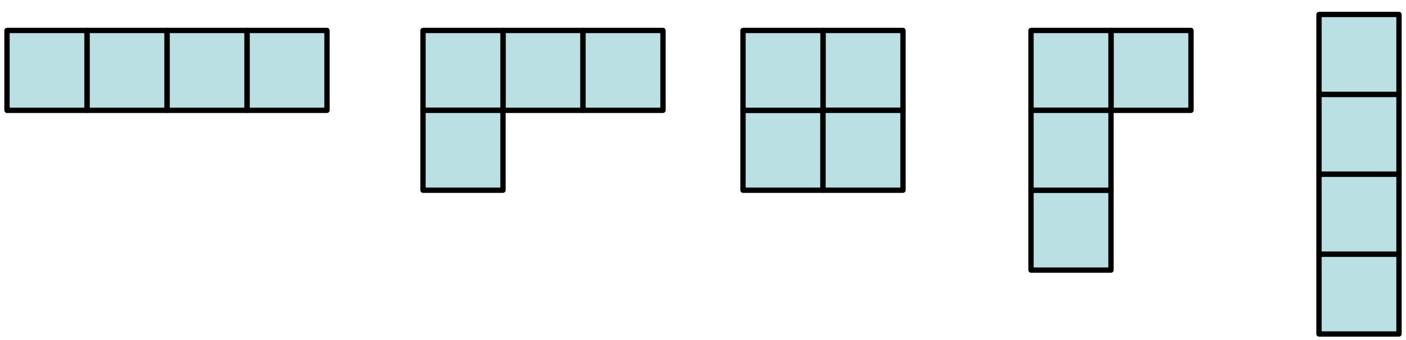
- Reduce “infinite” to “finite”
  – Set of n points in the plane: infinitely many
  – Poset patter only depends on ranking.
  – Thus, only depends on the permutation.
  点列の代わりに置換を考えよう
    - Finitely many, since the number of permutations of n points is n!
- If there is another set of n! elements, we may find a nice canonical one-to-one correspondence.

Young diagram

- Graphical representation of a partition of a number
  - (3), (2,1), (111)

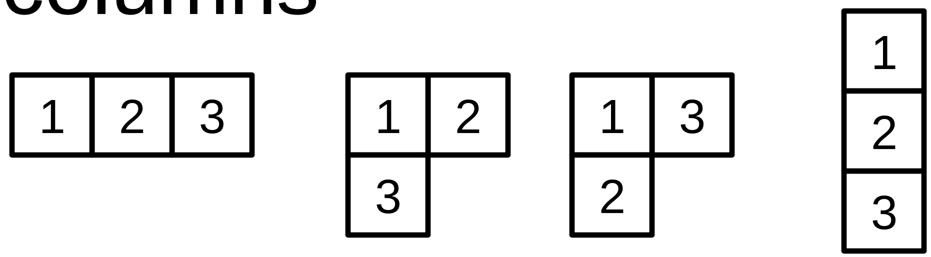


  - (4), (3,1), (2,2), (2,1,1),(1,1,1,1)



Young tableau(ヤング盤）

- Write 1 to n into diagram with increasing way in rows and columns
  - (3), (2,1), (111)

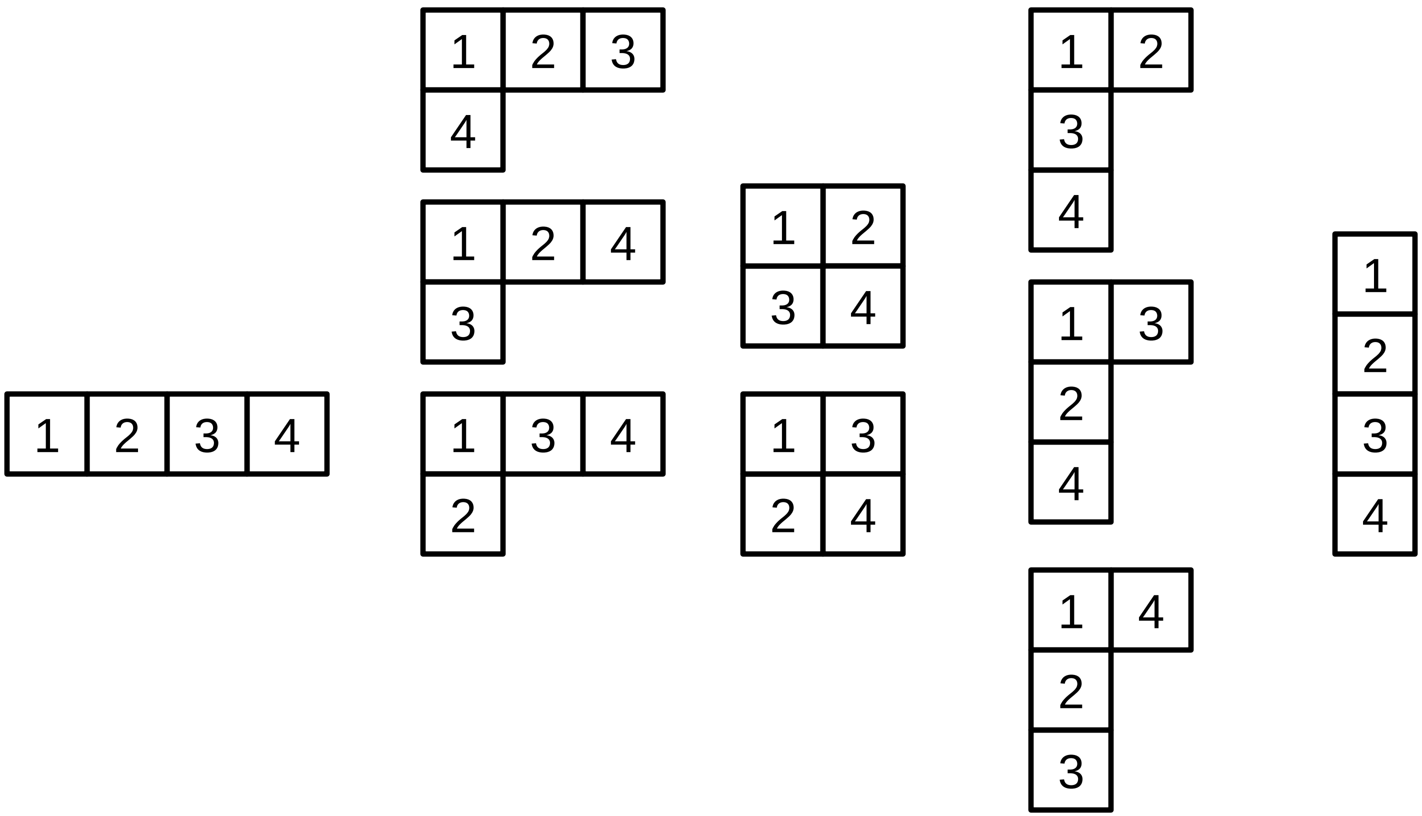


A strange relation: $1 + 2^2 + 1 = 6 = 3!$

Young tableau

(4), (3,1), (2,2), (2,1,1),(1,1,1,1)

$1 + 3^2 + 2^2 + 3^2 + 1 = 24 = 4!$



Robinson-Schensted-Knuth correspondence

- There is a canonical one-to one correspondence between permutations and pair of young tableaux of same shape

置換と、同じ形のYoung盤の間に一対一対応がある

  - Used in Group theory, Number theory, Algebraic Geometry, etc.

Algorithm

- Reference: Donald Knuth, The art of computer programming, Vol 3 sorting and searching. （Bible of computer science)
  - コンンピュータ科学の「聖書」のような本です
  - 日本語の本：寺田至 「ヤング図形のはなし」
- Algorithm will be given on the blackboard.
- Properties
  - The length of longest increasing (decreasing) subchain → width (height) of the diagram
  - If σ corresponds to (P(σ), Q(σ))、Then,
    $(P(\sigma^{-1}), Q(\sigma^{-1})) = (Q(\sigma), P(\sigma))$